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Discrete quadrature method for singular integrals on closed smooth contours.

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ABSTRACT

In this paper, a discrete quadrature structure is worked out for the numerical solution of a singular integral of the form

$$I(t_0) = \int_{\Omega} \frac{\psi(t)}{t - t_0} dt$$

where $t_0 \in \Omega$, $\psi(t)$ is smooth and belongs to the Holder's class $H(\alpha)$ on Ω. Ω is a closed smooth contour, which may be a standard circle of radius r or a closed Lyapunov contour. Some numerical results are obtained for the case of a unit circle with center at the origin.

Keywords: Discrete quadrature method, Singular integral, Lyapunov contour.

INTRODUCTION

Most of the functions considered here will be defined on smooth lines and will be assumed to satisfy the Holder condition (Muskhelishvili, 1972).

Throughout our discussions, only lines lying in one and the same plane will be considered. Unless otherwise stated, lines are always assumed to be simple, i.e. not to intersect.

Curves will be called smooth arcs, when they can be represented in the form:

$$x = x(s), \; y = y(s), \; s_a \le s \le s_b \qquad \dots (1.1)$$

where s_a, s_b are finite constants, x(s) and y(s) are functions continuous in the interval of definition, with the following properties:

1. They have continuous first derivatives $x'(s),\; y'(s)$ within the interval (s_a, s_b), including the end points, and these derivatives are never simultaneously zero: $x'(s),\; y'(s)$ at the ends of the interval are to be interpreted as $x'(s_a+0),\; y'(s_a+0)$ and $x'(s_b-0),\; y'(s_b-0)$ respectively.
2. The relation $x(s_1) \ne x(s_2)$ or $y(s_1) \ne y(s_1)$ holds for $s_a \le s_1$, $s_2 \le s_b$, $s_1 \ne s_2$.

Curves will be termed smooth contours, if they differ from smooth arcs only in that, in condition 2, the equalities hold iff $s_1 = s_2$; what is more, $x(s_b) = x(s_a)$, $y(s_b) = y(s_a)$ and $x'(s_b-0) = x'(s_a+0),\; y'(s_b-0) = y'(s_a+0)$.

The function $\psi(t)$ will be said to satisfy a Holder condition on Ω, if for any $t_1, t_2 \in \Omega$

$$|\psi(t_2) - \psi(t_1)| = A|t_2 - t_1|^{\alpha}, \qquad \dots (1.2)$$

where A and α are positive constants (Muskhelishvili, 1972). A is called the Holder constant and α the Holder index. Here, t denotes both the point t(x, y) and the corresponding complex number $t = x + iy$.

A function which satisfies a Holder condition will be said to obey the H condition or, when it is necessary to specify the index α, the $H(\alpha)$ condition. The value of the constant A is generally of no interest.

An arc is called Lyapunov (Shaposhnikova *et al*, 1975), if it has a well-defined tangent at every point and the angle θ between the tangents at the point t_1 and t_2 of this arc satisfies the inequality.

$$\theta \le A|t_1 - t_2|^{\alpha} \qquad \dots (1.3)$$

where A and α are the same as in (1.2) above.

DISCRETE QUADRATURE METHOD FOR A SINGULAR INTEGRAL $\int_{\Omega} \frac{dt}{t - t_0}$

In this paper, we consider the discrete quadrature formula for a singular integral

$$I(t_o) = \int_{\Omega} \frac{\psi(t)dt}{t - t_o} \qquad \dots (2.1)$$

where Ω is a unit circle with center at the origin, t_0 is an arbitrary point in Ω and $\psi(t)$ is a smooth function which belongs to the Holder class $H(\alpha)$ on Ω.

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It makes sense to begin with the integral

$$I_0(t_0) = \int_\Omega \frac{dt}{t-t_0} \quad \ldots \ (2.2)$$

where the value is already known (Muskhelishvili, 1972) and expressed as

$$I_0(t_0) = \pi i. \quad \ldots \ (2.3)$$

We choose on Ω, two sets of nodal points E = {t_k, k = $\overline{1,n}$} and E_o = {t_{ok}, k = $\overline{1,n}$} such that the points t_k, k = $\overline{1,n}$ divide the circle, Ω, into n different parts, while the points t_{0k} are chosen to be the mid-points of the arc $t_k t_{k-1}$. Notice that in this context, we have assumed $t_{n+1} = t_1$. In future, we shall refer to these sets E and E_o as the canonical subdivision of the circle Ω.

Lemma 1

For any point $t_{0j} \in E_0$, the following inequality is satisfied:

$$\left| \int_\Omega \frac{dt}{t-t_0} - \sum_{k=1}^{n} \frac{\Delta t_k}{t_k - t_{0j}} \right| \le 0\left(\frac{1}{n}\right) \quad \ldots \ (2.4)$$

where $\Delta t_k = t_{k+1} - t_k$, k = k = $\overline{1,n}$. The symbol $0\left(\frac{1}{n}\right)$ is used to represent quantities having the same order of singularity $\frac{1}{n}$, so that in the above inequality, the right-hand side can be thought of as a quantity $\frac{B}{n}$, where B is independent of n.

Since Ω is a unit circle and centred at the origin, we may write (Gandel, 1983; Noreddin and Tekhonenko, 1991)

$$t_k = e^{i\theta_k}, \quad t_{0k} = e^{i\theta_{0k}},$$

where θ_k and θ_{0k} are the polar angles of t_k and t_{0k} respectively, k = $\overline{1,n}$. Denoting $\eta_m = \frac{2\pi m}{n} - \frac{\pi}{n}, \quad m = 1, \ldots, n$

and considering the periodic nature of $e^{i\theta}$, we write

$$\sum_{k=1}^{n} \frac{\Delta t_k}{t_k - t_{0j}} = -\sum_{m=1}^{n} \frac{e^{i\eta_{m+1}} - e^{i\eta_m}}{1 - e^{i\eta_m}}$$

$$= \sum_{m=1}^{n} \left[\cot\frac{\eta_m}{2} \cos\frac{\Delta\eta_m}{2} - \sin\frac{\Delta\eta_m}{2} + \right.$$

$$\left. + \ i\left(\cos\frac{\Delta\eta_m}{2} + \cot\frac{\Delta\eta_m}{2} \sin\frac{\Delta\eta_m}{2} \right) \right] \sin\frac{\Delta\eta_m}{2} \quad \ldots \ (2.5)$$

where $\Delta\eta_m = \eta_{m+1} - \eta_m = \frac{\eta_m}{2}, \quad m = 1, \ldots, n.$

Notice that since the numbers $\frac{\eta_m}{2}$, m = 1, …, n are symmetrically distributed about the $\frac{\pi}{2}$ line, the equality

$$\sum_{m=1}^{n} \cot\frac{\eta_m}{2} = 0 \quad \ldots \ (2.6)$$

is valid.

From equations (2.5) and (2.6), it follows that

$$\sum_{m=1}^{n} \frac{\Delta t_k}{t_k - t_{0j}} = -n\sin^2\frac{\pi}{2} + i\frac{n}{2}\sin\frac{2\pi}{n} = i\pi + 0\left(\frac{1}{n}\right) \quad \ldots \ (2.7)$$

Thus, together with equation (2.3), the validity of relation (2.4) is proved.

Remark 1

The following estimate is valid:

$$\sum_{k=1}^{n} \frac{|\Delta t_k|}{|t_{0j} - t_k|} \le O(\ln n), \quad j = 1, \ldots, n \quad \ldots \ (2.8)$$

Indeed, it is observed that

$$\sum_{k=1}^{n} \frac{\Delta t_k}{t_k - t_{0j}} = \sum_{m=1}^{n} \frac{\left|\sin\frac{\Delta\eta_m}{2}\right|}{\left|\sin\frac{\eta_m}{2}\right|} \le C \sum_{m=1}^{\left[\frac{n}{2}\right]+1} \frac{\Delta\eta_m}{2} \cdot \frac{2}{\eta_m} = O(\ln n) \quad \ldots \ (2.9)$$

where [x] is the whole part of the number x.

THE METHOD

Let us now attempt to put together a similar quadrature structure for the singular integral in (2.1) (Hermann, 1990; Bialecki *et al*, 2004). We shall, in this context, still consider the sets E and E_o as the canonical subdivision of the circle Ω. Let

$$S_n(t_{0j}) = \sum_{k=1}^{n} \frac{\psi(t_k)\Delta t_k}{t_k - t_{0j}}, \quad j = 1, \ldots, n. \quad \ldots \ (3.1)$$

The following theorem is valid.

Theorem 1

Let ψ(t) obey the H(α) condition on Ω. Then the following inequality is satisfied:

$$\left| I(t_{0j}) - S_n(t_{0j}) \right| \le \theta(t_{0j}), \quad j = 1, \ldots, n \quad \ldots \ (3.2)$$

where $\theta(t_{0j}) = O\left(\frac{1}{n^\alpha}\ln n\right) + \left|\psi(t_{0j})\right| O\left(\frac{1}{n}\right)$.

Proof:

For convenience, we shall take $t_{oj} = 1$. In this case,

$$\left| I(t_{0j}) - S_n(t_{0j}) \right| \le I_1 + I_2.$$

$$I_1 = \left| \int_\Omega \frac{\psi(t) - \psi(1)}{t-1} dt - \sum_{k=1}^{n} \frac{\psi(t_k) - \psi(1)}{t_k - 1} \Delta t_k \right|, I_2 = |\psi(1)| \left| \int_\Omega \frac{dt}{t-1} - \sum_{k=1}^{n} \frac{\Delta t_k}{t_k - 1} \right|.$$

Relation (2.4) immediately provides the estimate for I_2. For expression I_1, we further rearrange as follows:

$$I_1 \quad \leq \quad I_1' + I_1'' + I_1'''$$

where

$$I_1' = \left| \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \left[\frac{\psi(t)-\psi(1)}{t-1} - \frac{\psi(t_k)-\psi(1)}{t_k-1} \right] dt \right|, I_1'' = \left| \int_{t_n}^{t_1} \frac{\psi(t)-\psi(1)}{t-1} dt \right|,$$

and

$$I_1''' \quad = \quad \left| \frac{\psi(t_n)-\psi(1)}{t_n-1} \right| \frac{2\pi}{n}.$$

Since ψ(t) belongs to Holder's class in Ω,

$$I_1'' \quad \leq \quad \int_{t_n}^{t_1} \frac{|\psi(t)-\psi(1)|}{|t-1|} |dt| \quad \leq \quad A \int_{t_n}^{t_1} |t-1|^{-1+\alpha} |dt|.$$

For a unit circle, |dt| = dθ and

$$|t-1| \quad = \left| e^{i\theta} - 1 \right| \quad = \left| (\cos\theta - 1) + i \sin\theta \right| \quad = 2 \left| \sin \frac{\theta}{2} \right|.$$

Therefore,

$$I_1'' \leq A 2^\alpha \int_0^{\frac{\pi}{n}} \left(\sin \frac{\theta}{2} \right)^{-1+\alpha} d\theta \quad \leq C_1 \int_0^{\frac{\pi}{n}} \theta^{-1+\alpha} d\theta = O\left(\frac{1}{n^\alpha} \right).$$

For I_1''', we have

$$I_1''' \quad \leq \quad \frac{2\pi}{n} A |t_n - 1|^{-1+\alpha} \quad = \quad O\left(\frac{1}{n^\alpha} \right).$$

However, the estimate for I_1' will undergo some kind of further simplification with the understanding that we shall require its services in future. Indeed,

$$\frac{\psi(t)-\psi(t_{0j})}{t-t_{0j}} - \frac{\psi(t_k)-\psi(t_{0j})}{t_k-t_{0j}}$$

$$= \quad \frac{\psi(t)-\psi(t_k)}{t-t_{0j}} + \frac{[\psi(t_k)-\psi(t_{0j})](t_k-t)}{(t-t_{0j})(t_k-t_{0j})} \qquad \dots \ (3.3)$$

Consequently,

$$I_1' \leq \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \frac{|\psi(t)-\psi(t_k)|}{|t-1|} |dt| + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \frac{|\psi(t_k)-\psi(1)|}{|t_k-1|} \cdot \frac{|t_k-t|}{|t-1|} |dt| \quad = S_1 + S_2.$$

Since ψ(t) belongs to Holder's class in Ω and t = e$^{i\theta}$, we have

$$S_1 \leq A \left(\frac{2\pi}{n} \right)^\alpha \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \frac{|dt|}{|t-1|} \leq C_2 \frac{1}{n^\alpha} \int_{\frac{n}{\pi}}^{\frac{\pi}{2}} \quad = O\left(\frac{\ln n}{n^\alpha} \right) \frac{d\theta}{\theta}$$

Finally, for the second sum, we have

$$S_2 \leq A \frac{2\pi}{n} \sum_{k=1}^{n-1} \frac{1}{|t_k - 1|^{1-\alpha}} \int_{t_k}^{t_{k+1}} \frac{|dt|}{|t-1|} \leq C_3 \frac{\pi}{n} \int_{\frac{\pi}{n}}^{\pi} \frac{d\theta}{\theta^{2-\alpha}} = O\left(\frac{1}{n^\alpha} \right)$$

These last expressions for S_1 and S_2 provide the estimate for I_1'. Thus, together with I_1'' and I_1''', it is clear that

$$I_1 \quad \leq \quad O\left(n^{-\alpha} \ln n \right).$$

This proves theorem 1.

Definition

A function φ(t) is said to belong to the class Π on Ω if it can be expressed in the form

$$\varphi(t) \quad = \quad \frac{\psi(t)}{q-t}$$

where ψ(t) ∈ H(α) on Ω and q is some fixed point on Ω.

Notice that we can write

$$\int_\Omega \frac{\varphi(t)}{t-t_0} dt = \frac{1}{q-t_0} \left[\int_\Omega \frac{\psi(t)}{t-t_0} dt - \int_\Omega \frac{\psi(t)}{t-q} dt \right].$$

This form of representation of the original singular integral guarantees the validity of the following theorem.

Theorem 2

Let φ(t) ∈ Π on the circle Ω, and let E and E_o be the canonical subdivision of Ω, what is more, let q ∈ E_o for j = j_q. Then the inequality

$$\left| I(t_{0j}) - S_n(t_{0j}) \right| \leq \theta(t_{0j}), \quad j \neq j_q, \quad j = 1, \dots, n \qquad \dots \ (3.4)$$

is valid, where θ(t_{0j}) has the form

$$\theta(t_{0j}) \quad = \quad \frac{1}{|t_{0j} - q|} O\left(\frac{1}{n^\alpha} \ln n \right), \quad j \neq j_q.$$

Clearly, θ(t_{0j}) satisfies the relations

$$\theta(t_{0j}) \quad \leq \quad O_l\left(\frac{1}{n^{\lambda_1}} \right), \quad \lambda_1 > 0 \qquad \dots \ (3.5)$$

for all t_{0j} ∈ Ω*l*, where *l* is an ε - neighbourhood of q; ε being a very small pre-assigned positive number, and

$$\sum_{\substack{j=1 \\ j \neq j_q}}^{n} \theta(t_{0j}) |\Delta t_{0j}| \quad \leq \quad O\left(\frac{1}{n^{\lambda_2}} \right), \quad \lambda_2 > 0. \qquad \dots \ (3.6)$$

Notice that (3.5) can be written as $O_l\left(n^{-\alpha} \ln n\right)$ and (3.6) in the form of $O\left(n^{-\alpha} \ln n\right)$.

Remark 2

Relation (3.2) still remains valid for the integral $\int_\Omega \frac{\varphi(t,t_0)}{t-t_0} dt$ if φ(t,t_0) ∈ H(α) in Ω, that is,

$$\left| \int_\Omega \frac{\varphi(t,t_{0j})}{t-t_{0j}} dt - \sum_{k=1}^{n} \frac{\varphi(t_k,t_{0j})}{t_k - t_{0j}} \right| \leq O\left(\frac{\ln n}{n^\alpha} \right) \qquad \dots \ (3.7)$$

Remark 3

Now, let Ω_1 be a closed Lyapunov contour. Then between the points τ of this curve and the points t of a standard circle L (in particular, a unit circle with center at the origin), there exists a one-to-one correspondence τ = τ (t) such that $\tau'(t) = \frac{d\tau}{dt}$ belongs to H(α) and does not vanish anywhere on Ω. Now if φ(τ) ∈ H(α) on Ω_1, then by virtue of the formula for change of variable in singular integral (Muskhelishvili, 1972), we obtain

$$\int_{\Omega_1} \frac{\varphi(\tau)}{\tau - t_0} d\tau = \int_{\Omega} \frac{\psi(t,t_0)}{t - t_0} dt, \quad \psi(t,t_0) = \frac{t - t_0}{\tau(t) - \tau(t_0)} \tau'(t)\varphi(\tau(t)).$$

We call the set of points $\tau_k = \tau(t_k)$ ($t_k \in E$) and $\tau_{ok} = \tau(t_{ok})$ ($t_{ok} \in E_o$) the canonical subdivision of the contour Ω_1. Let us consider the sum

$$\sum_{k=1}^{n} \frac{\varphi(\tau_k)\Delta\tau_k}{\tau_k - \tau_{0j}} = \sum_{k=1}^{n} \frac{\varphi(\tau(t_k))(t_k - t_{0j})}{\tau(t_k) - \tau(t_{0j})} \cdot \frac{\Delta\tau_k}{\Delta t_k} \cdot \frac{\Delta t_k}{t_k - t_{0j}}.$$

Since,

$$\frac{\Delta\tau_k}{\Delta t_k} = \frac{\tau(t_{k+1}) - \tau(t_k)}{t_{k+1} - t_k} = \tau'(\tilde{t}_k), \quad \tilde{t}_k \in arc\,(t_k t_{k+1}),$$

and $\tau'(t) \in H(\beta)$ on Ω, then

$$\left|\tau'(\tilde{t}_k) - \tau'(t_k)\right| \le A\left|\tilde{t}_k - t_k\right|^{\beta},$$

that is,

$$\tau'(\tilde{t}_k) = \tau'(t_k) + O\left(\left|\tilde{t}_k - t_k\right|^{\beta}\right).$$

Consequently,

$$\sum_{k=1}^{n} \frac{\varphi(\tau_k)}{\tau_k - \tau_{0j}} \Delta\tau_k = \sum_{k=1}^{n} \frac{\varphi(\tau(t_k))(t_k - t_{0j})}{\tau(t_k) - \tau(t_{0j})} \bullet \frac{\tau'(t_k)\Delta t_k}{t_k - t_{0j}}$$
$$+ \sum_{k-1}^{n} \frac{\varphi(\tau(t_k))(t_k - t_{0j})}{\tau(t_k) - \tau(t_{0j})} O\left(\left|\tilde{t}_k - t_k\right|^{\beta}\right) \frac{\Delta t_k}{t_k - t_{0j}}$$
$$= S_j^1 + S_j^2.$$

Using (2.8), it is immediately clear that

$$\left|S_j^2\right| \le O\left(n^{-\beta} \ln n\right).$$

Therefore from (3.7) and the estimate for S_j^2, we obtain the inequality

$$\left|\int_{\Omega_1} \frac{\varphi(\tau)}{\tau - \tau_{0j}} dr - \sum_{k=1}^{n} \frac{\varphi(\tau_k)}{\tau_k - \tau_{0j}} \Delta\tau_k\right| \le O\left(\frac{1}{n^{\lambda}}\right), \quad \lambda > 0. \quad \ldots (3.8)$$

Now let Ω represent the union of p non-intersecting closed Lyapunov curves $\Omega_1, \ldots, \Omega_p$ and let the sets $E_m = \{\tau_k, k = n_{m-1}+1, \ldots, n_m\}$ and let $E_{om} = \{\tau_{ok}, k = n_{m-1}+1, \ldots, n_m\}$ be the canonical subdivision of Ω_m, m = 1, …, p into $N_m = n_m - n_{m-1}$, where we have assumed that $n_0 = 0$. Let $N = \min_{m=1,\ldots,p} N_m$. In future, we shall assume that $N_m / N \le R < +\infty$. Again, let

$$S_{n_p}(\tau_{0j}) = \sum_{k=1}^{n_p} \frac{\varphi(\tau_k)\Delta\tau_k}{\tau_k - \tau_{0j}}, \quad j = 1, \ldots, n_p$$

where $\Delta\tau_k = \tau_{k+1} - \tau_k$, $k = 1, \ldots, n_p$, $k \ne n_1, \ldots, n_p$ and $\Delta\tau_{n_m} = \tau_{n_m+1} - \tau_{n_m}$, $m = 1, \ldots, p$. The following theorem holds:

Theorem 3

Let $\varphi(\tau) \in H(\lambda)$ on Ω. Then for any $\tau_{0j} \in \bigcup_{m=1}^{p} E_{0m}$, the inequality

$$\left|I(\tau_{0j}) - S_{n_p}(\tau_{0j})\right| \le O\left(\frac{1}{N^{\lambda}}\right), \quad \lambda > 0$$

is valid.

APPENDIX

In this section, we consider the numerical integration of the singular integral

$$I(t_a) = \int_{\Omega} \frac{\sqrt{1-t^2}}{t_0 - t} dt$$

where Ω is a unit circle centred at the origin and defined in the complex plane C; t and t_0 being points of the circle Ω. The numerical results, based on equal partitioning of Ω, was written in Java and is as shown in the table below.

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Table 1. Numerical results, based on equal partitioning of Ω

i	0	1	2	3	4
t_i	1.00+0.00i	0.00+0.16i	0.95+0.31i	0.89+0.45i	0.81+0.59i
$S_{40}(t_{0j})$	-0.06-0.07i	-0.13- 0.17i	-0.21- 0.29i	-0.29- 0.45i	-0.36- 0.62i

i	5	6	7	8	9
t_i	0.71+0.71i	0.59+0.81i	0.45+0.89i	0.31+0.95i	0.16+0.99i

i	10	11	12	13	14
t_i	0.00+1.00i	-0.16+0.99i	-0.31+0.95i	-0.45+0.89i	-0.59+0.81i
$S_{40}(t_{0j})$	-0.50-1.89i	-0.47- 2.11i	-0.42- 2.31i	-0.36- 2.50i	-0.29- 2.67i

$S_{40}(t_{0j})$	-0.42-0.81i	-0.42- 1.01i	-0.50- 1.23i	-0.52- 1.45i	-0.52- 1.67i

Table 1 continued**. Numerical results, based on equal partitioning of Ω**

i	15	16	17	18	19
t_i	-0.71+0.71i	-0.81+0.59i	-0.89+0.45i	-0.95+0.31i	-0.99+0.16i
$S_{40}(t_{0j})$	-0.21- 2.83i	-0.13- 2.95i	-0.06- 3.05i	0.00- 3.12i	0.00- 3.12i
i	20	21	22	23	24
t_i	-1.00+0.00i	-0.99-0.16i	-0.95-0.31i	-0.89-0.45i	-0.81-0.59i
$S_{40}(t_{0j})$	-0.06- 3.19i	-0.13- 3.29i	-0.21- 3.42i	-0.29- 3.57i	-0.36- 3.74i
i	25	26	27	28	29
t_i	-0.71-0.71i	-0.59-0.81i	-0.45-0.89i	-0.31-0.95i	-0.16-0.99i
$S_{40}(t_{0j})$	-0.42-3.93i	-0.47- 4.14i	-0.50- 4.35i	-0.52- 4.57i	-0.52- 4.79i
i	30	31	32	33	34
t_i	0.00-1.00i	0.16-0.99i	0.31-0.95i	0.45-0.89i	0.59-0.81i
$S_{40}(t_{0j})$	-0.50-5.01i	-0.47-5.23i	-0.42-5.43i	-0.36-5.62i	-0.29-5.80i
i	35	36	37	38	39
t_i	0.71-0.71i	0.81-0.59i	0.89-0.45i	0.95-0.31i	0.99-0.16i
$S_{40}(t_{0j})$	-0.21-5.95i	-0.13-6.08i	-0.06-6.18i	0.00- 6.24i	0.00- 6.24i

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